

Friday, November 7, Lecture 24: Fourier transform & Spectral theory

Last time... $\mathcal{H} = L^2([-\pi, \pi])$ or $L^2(\mathbb{Z})$.

- Spectral Thm: $A = A^\dagger$. Then A can be unitarily diagonalized:

$$A = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^\dagger$$

- Multiplication operators $M_V: \mathcal{H} \rightarrow \mathcal{H}$, $V \in \mathcal{H}$, are like diagonal matrices:

$$(M_V f)(x) = V(x) f(x)$$

• If $v: \{1, 2, \dots, n\} \rightarrow \mathbb{C}$, then

$$M_v = \begin{bmatrix} v(1) & & \\ & \ddots & \\ & & v(x) & \\ & & & \ddots \\ & & & & v(n) \end{bmatrix}, \quad f = \begin{bmatrix} f(1) \\ \vdots \\ f(x) \\ \vdots \\ f(n) \end{bmatrix}$$

then

$$(M_v f)(x) = v(x) f(x), \quad x = 1, \dots, n$$

- If $\overline{V(x)} = V(x)$, $M_v = M_v^\dagger$ is self-adjoint

Ex: Diagonal matrix $D = D^\dagger$ if entries are real:

$$D^\dagger = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^\dagger = \begin{pmatrix} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_n \end{pmatrix} \stackrel{\text{if } \lambda_i \in \mathbb{R}}{=} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = D$$

- Derivative operator: $p = -i \frac{d}{dx}$ is self-adjoint,

$$p = p^\dagger.$$

So: how do we find basis of eigenvectors for p ?

Observe:

$$\underbrace{-i \frac{d}{dx}}_P e^{ikx} = k e^{ikx}$$

$$\underbrace{P}_P v_k = k v_k !$$

So: Fourier series gives eigenbasis for $-i \frac{d}{dx}$!

Observe:

$$\begin{aligned} -i \frac{d}{dx} f(x) &= -i \frac{d}{dx} \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx} \\ &= \sum_{k=-\infty}^{\infty} \hat{f}(k) \left(-i \frac{d}{dx}\right) e^{ikx} \\ &= \sum_{k=-\infty}^{\infty} \hat{f}(k) k e^{ikx} \end{aligned}$$

Thm: (Fourier basis diagonalizes derivatives)

$$\widehat{\left(-i \frac{d}{dx} f\right)}(k) = k \hat{f}(k) \quad (*)$$

• Why do we call this diagonalization?

• Parseval: $f \mapsto \hat{f} = Uf$ is unitary

$$\begin{array}{ccc} L^2([- \pi, \pi]) & \xrightarrow{U} & l^2(\mathbb{Z}) \\ & \xleftarrow{U^+} & \\ U^+ \hat{f} = f & \xleftarrow{\quad} & \hat{f} \end{array}$$

• The self-adjoint operator is

$$p = -i \frac{\partial}{\partial x}.$$

$$(*) \quad (U p U^+ \cdot \hat{f})(k) = (M_k \hat{f})(k) = k \cdot \hat{f}(k)$$

or $U p U^+ = M_k \leftarrow \text{diagonal operator!}$

Another perspective: unitaries map orthonormal bases to orthonormal bases.

$$f \in L^2([-\pi, \pi]), \text{ periodic boundary conditions} \begin{array}{c} \xrightarrow{U} \\ \xleftarrow{U^\dagger} \end{array} \hat{f} \in \ell^2(\mathbb{Z})$$

onbs: $\{e^{ikx} : k \in \mathbb{Z}\}$

$\{\delta_k : k \in \mathbb{Z}\}$

$$1 \longleftrightarrow \delta_0 = (\dots, 0, 1, 0, 0, 0, \dots)$$

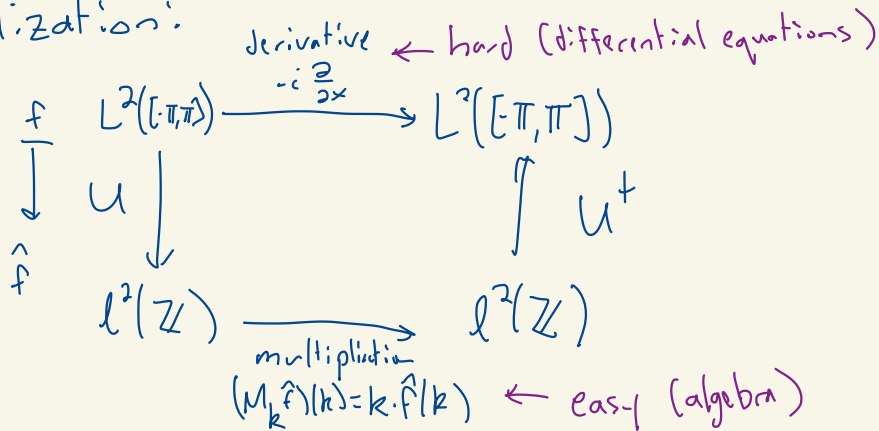
$$e^{ix} \longleftrightarrow \delta_1 = (\dots, 0, 0, 1, 0, 0, \dots)$$

$$e^{i2x} \longleftrightarrow \delta_2 = (\dots, 0, 0, 0, 1, 0, \dots)$$

$$e^{i3x} \longleftrightarrow \delta_3 = (\dots, 0, 0, 0, 0, 1, \dots)$$

-

Diagonalization:



To highlight the power of this approach, return to the heat equation on circle:

Temperature

$$T: [-\pi, \pi] \times [0, \infty) \rightarrow \mathbb{R}$$

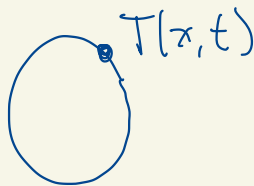
$\uparrow$
coordinate
on circle

$\uparrow$
time

satisfies

$$\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T \leftarrow \text{heat eqn.}$$

$$T(x, 0) = f(x) \leftarrow \text{initial data}$$



Look for solution (separation of variables)

$$T(x, t) = \sum_{k=-\infty}^{\infty} T_k(t) e^{-ikx}$$

$\underbrace{\hspace{10em}}_{\text{Fourier coeffs depend on } t}$

$\leftarrow$ so, T is now represented in eigenbasis of $\frac{\partial^2}{\partial x^2}$!

Heat eqn for each coef:

$$\left[\begin{array}{l} \frac{\partial T_k}{\partial t} + k^2 T_k(x, t) = 0 \\ T_k(x, 0) = \hat{f}(k) \end{array} \right]$$

$\leftarrow$ Diagonalization!

This is easy to solve.

$$T_k(t) = \hat{f}(k) e^{-k^2 t}$$

So the full solution is

$$T(x,t) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{-k^2 t} e^{ikx}$$

A few observations:

Fast oscillations



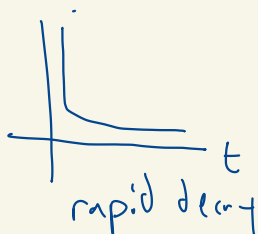
k big

Slow oscillations

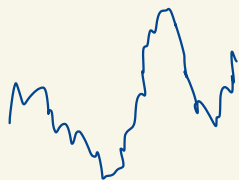


k small

$e^{-k^2 t}$



$\Rightarrow$ fast oscillations decay quickly,
which leads to smoothing of initial data:



$T(x, 0)$



$T(x, t_{\text{small}})$



$T(x, t_{\text{big}})$

- Exponential decay of solutions to equilibrium:

Split

$$T(x,t) = \hat{f}(0) + \sum_{k \neq 0} \hat{f}(k) \underbrace{e^{-k^2 t}}_{\substack{\uparrow \text{this term} \\ \text{drives to 0}}} e^{ikx}$$

$\uparrow$
 constant term

$$\hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx, \text{ average value}$$

This approach is very useful to study several PDEs!